Solvable few-body quantum problems

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Abstract :

This work is devoted to the study of some exactly solvable quantum problems of four, five and six bodies moving on the line. We solve completely the corresponding stationary Schrödinger equation for these systems confined in an harmonic trap, and interacting pairwise, in clusters of two and three particles, by two-body inverse square Calogero potential. Both translationaly and non-translationaly invariant multi-body potentials are added. In each case, the full solutions are provided, namely the normalized regular eigensolutions and the eigenenergies spectrum. The irregular solutions are also studied. We discuss the domains of coupling constants for which these irregular solutions are square integrable. The case of a "Coulomb-type" confinement is investigated only for the bound states of the four-body systems.

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1 Introduction

A limited number of exactly solvable many-body systems exists, in the one dimensional space [1, 2]. The Calogero model constitutes a famous example, which was exhaustively studied [3, 4]. A survey of quantum integrable systems was done by Olshanetsky and Perelomov [5]. They classified the systems with respect to Lie algebras. Point interactions have also been considered, still in $D = 1$ dimensional space [6, 7].

The search for exactly solvable few-body systems concerns mostly the three-body case and still retains attention. Early works of three-body linear problems of Calogero-Marchioro-Wolfes [8, 9, 10] have been followed by new extensions and cases. In a non exhaustive way, we can quote the three-body problems of Khare *et al.* [11], Quesne [12] and Meljanac *et al.* [13]. In a recent paper, we have exactly solved a generalization of the three-body Calogero-Marchioro-Wolfes problem with an additional non-translationally three-body potential [14].

Works on the four-body problem and beyond are much more scarce [15, 16]. This is due to the difficulty to extend right away the Calogero model to a larger number of particles. We can quote the work of Haschke and Rühl [17] concerning the construction of exactly solvable quantum models of Calogero and Sutherland type with translationally invariant two-and four particles interactions.

Recently an extension of the four-body problem of Wolfes [15] with non-translationally invariant interactions has been investigated and solved exactly [18]. The purpose of the present work is to point out particular cases admitting an exact solution. In the four-body problem, we consider the particles to be experiencing either an harmonic field or to be submitted to an attractive $1/r$-type potential. The interaction among the particles is inspired by the one used in our previous work on the three-body potential [14]. It contains two- and many-body forces.

The paper is organized as follows. In section **2** we expose and solve the four-body problem for the case of both harmonic and "Coulomb-type" confinement of the particles. Section **3** and **4** are devoted to the solutions of similar five- and six-body problems respectively. Conclusions are drawn in section **5**.

2 A four-body problem

2.1 The case with harmonic confinement.

We consider the Hamiltonian

$$H = \sum_{i=1}^{4} \left(-\frac{\partial^2}{\partial x_i^2} + \omega^2 x_i^2 \right) + \lambda \sum_{i<j}^{3} \frac{1}{(x_i - x_j)^2} + \frac{g}{(x_1 + x_2 + x_3 - 3x_4)^2} + \frac{\mu}{\sum_{i=1}^{4} x_i^2} \tag{1}$$

Here, we use the units $\hbar = 2m = 1$. The first contribution gives the energy of four independent particles in an harmonic field. To this part, residual interactions are added in the following way. The first three particles, with coordinates x_1, x_2 and x_3, interact pairwise by a two-body inverse square potentials, of Calogero type [8]. Their centre of mass interacts with the fourth particle, whose the coordinate is x_4, via the translationally invariant term $g/(x_1 + x_2 + x_3 - 3x_4)^2$. An additional non translationally invariant four-body potential, represented by the term $\mu/(x_1^2 + x_2^2 + x_3^2 + x_4^2)$, is added to the whole Hamiltonian.

In order to solve this four-body problem, let us introduce the following coordinates transformation

$$t = \frac{1}{2}(x_1 + x_2 + x_3 + x_4), \quad u = \frac{1}{\sqrt{2}}(x_1 - x_2), \quad v = \frac{1}{\sqrt{6}}(x_1 + x_2 - 2x_3), \quad w = \frac{1}{2\sqrt{3}}(x_1 + x_2 + x_3 - 3x_4).$$

(2)

The transformed Hamiltonian reads:

$$\begin{aligned} H &= -\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial w^2} - \frac{\partial^2}{\partial u^2} - \frac{\partial^2}{\partial v^2} + \omega^2(t^2 + u^2 + v^2 + w^2) + \frac{\mu}{t^2 + u^2 + v^2 + w^2} \\ &\quad + \frac{9\lambda(u^2 + v^2)^2}{2\,(u^3 - 3uv^2)^2} + \frac{g}{12w^2}. \end{aligned}$$

(3)

This Hamiltonian is not separable in $\{t, u, v, w\}$ variables. To overcome this situation we introduce the following hyperspherical coordinates :

$$\begin{aligned} t &= r\cos\alpha, \quad w = r\sin\alpha\cos\theta, \quad u = r\sin\alpha\sin\theta\sin\varphi, \quad v = r\sin\alpha\sin\theta\cos\varphi, \\ 0 &\leq r < \infty, \quad 0 \leq \alpha \leq \pi, \quad 0 \leq \theta \leq \pi, \quad 0 \leq \varphi \leq 2\pi. \end{aligned}$$

(4)

The Schrödinger equation is then written:

$$\begin{aligned} &\left\{ -\frac{\partial^2}{\partial r^2} - \frac{3}{r}\frac{\partial}{\partial r} + \omega^2 r^2 + \frac{\mu}{r^2} + \frac{1}{r^2}\left[-\frac{\partial^2}{\partial \alpha^2} - 2\cot\alpha\frac{\partial}{\partial \alpha} + \frac{1}{\sin^2\alpha}\left(-\frac{\partial^2}{\partial \theta^2} - \cot\theta\frac{\partial}{\partial \theta} \right.\right.\right. \\ &\quad \left.\left.\left. + \frac{g}{12\cos^2\theta} + \frac{1}{\sin^2\theta}\left(-\frac{\partial^2}{\partial \varphi^2} + \frac{9\lambda}{2\sin^2(3\varphi)} \right) \right) \right] \right\} \Psi(r, \alpha, \theta, \varphi) = E\Psi(r, \alpha, \theta, \varphi) \,. \end{aligned}$$

(5)

This Hamiltonian may be mapped to the problem of one particle in the four dimensional space with a non central potential of the form

$$V(r, \alpha, \theta, \varphi) = f_1(r) + \frac{1}{r^2\sin^2\alpha}\left[f_2(\theta) + \frac{f_3(\varphi)}{\sin^2\theta} \right].$$

(6)

It is then clear that the problem becomes separable in the four variables $\{r, \alpha, \theta, \varphi\}$. To find the solution we factorize the wave function as follows :

$$\Psi_{k,\ell,m,n}(r, \alpha, \theta, \varphi) = \frac{F_{k,\ell,m,n}(r)}{r\sqrt{r}}\frac{G_{\ell,m,n}(\alpha)}{\sin\alpha}\frac{\Theta_{m,n}(\theta)}{\sqrt{\sin\theta}}\Phi_n(\varphi).$$

(7)

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Accordingly, equation (5) separates in four decoupled differential equations:

$$\left(-\frac{d^2}{d\varphi^2} + \frac{9\lambda}{2\sin^2(3\varphi)}\right)\Phi_n(\varphi) = B_n\Phi_n(\varphi), \tag{8}$$

$$\left(-\frac{d^2}{d\theta^2} + \frac{(B_n - \frac{1}{4})}{\sin^2\theta} + \frac{g}{12\cos^2\theta}\right)\Theta_{m,n}(\theta) = C_{m,n}\Theta_{m,n}(\theta), \tag{9}$$

$$\left(-\frac{d^2}{d\alpha^2} + \frac{C_{m,n} - \frac{1}{4}}{\sin^2\alpha}\right)G_{\ell,m,n}(\alpha) = D_{\ell,m,n}G_{\ell,m,n}(\alpha), \tag{10}$$

and

$$\left(-\frac{d^2}{dr^2} + \omega^2 r^2 + \frac{\mu + D_{\ell,m,n} - \frac{1}{4}}{r^2}\right)F_{k,\ell,m,n}(r) = E_{k,\ell,m,n}\,F_{k,\ell,m,n}(r)\;. \tag{11}$$

The potential of Eq.(8) has a periodicity of $\frac{\pi}{3}$. In the interval $0 \leq \varphi \leq 2\pi$, it possesses singularities at $\varphi = k\frac{\pi}{3}, k = 0, 1, ..., 5$. This equation has been solved by Calogero [8]. In the vicinity of $\varphi = 0$ (resp. $\frac{\pi}{3}$,) the singularity can be treated if and only if $\lambda > -1/2$, similar to the case of a centrifugal barrier. Otherwise the operator has several self-adjoint extensions, each of which may lead to a different spectrum [19, 20].

The solutions of equation (8) on the interval $[0, \pi/3]$, with Dirichlet conditions at the boundaries, in terms of orthogonal polynomials, are given by [8, 14]

$$\Phi_n^{(\pm)}(\varphi) = (\sin 3\varphi)^{\frac{1}{2}\pm a}C_n^{(\frac{1}{2}\pm a)}(\cos 3\varphi), \quad 0 \leq \varphi \leq \frac{\pi}{3}, \qquad n = 0, 1, 2, ... \tag{12}$$

The $C_n^{(q)}$ denote the Gegenbauer polynomials [21]. The corresponding eigenvalues take the form

$$B_n^{(\pm)} = 9\left(n + \frac{1}{2} \pm a\right)^2, \qquad n = 0, 1, 2, ... \tag{13}$$

with

$$a = \frac{1}{2}\sqrt{1 + 2\lambda}\;. \tag{14}$$

The extension to the whole interval $[0, 2\pi]$ is achieved following the prescription given in [8], by using symmetry arguments according to the statistics obeyed by the particles. Generally, only the regular solution, $\Phi_n^{(+)}$, corresponding to $1/2 + a$, is retained. However, the irregular solution, $\Phi_n^{(-)}$, which is distinct of $\Phi_n^{(+)}$ for $\lambda > -1/2$, is physically acceptable when the Dirichlet condition is satisfied for $-1/2 < \lambda \leq 0$ (attractive potentials). If we release the Dirichlet condition, and ask only for the square integrability of the solution, as in [22], then $\Phi_n^{(-)}$ can be retained for $-1/2 < \lambda < 3/2$ [14]. For $\lambda = 0$, the pairwise interaction between the particles $1, 2$ and 3 disappears.

The second angular equation for the polar angle θ can be written as

$$\left(-\frac{d^2}{d\theta^2} + \frac{b_n^2 - \frac{1}{4}}{\sin^2\theta} + \frac{\beta}{\cos^2\theta} - C_{m,n}\right)\Theta_{m,n}(\theta) = 0, \tag{15}$$

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where the auxiliary constants b_n, β are defined respectively by

$$b_n^2 = B_n, \qquad b_n = \sqrt{B_n} = \left(3n + 3a + \frac{3}{2} \right). \tag{16}$$

and

$$\beta = \frac{g}{12} . \tag{17}$$

Since $b_n \neq 0$, the Hamiltonian of Eq.(15) is a self-adjoint operator provided that $\beta > -1/4$.

Considering the interval $0 < \theta < \pi$, we first note that Eq.(15) has three singularities occurring at $\theta = k\frac{\pi}{2}$, with $k = 0, 1, 2$. This separates the interval $]0, \pi[$ in two equal length intervals namely

$$0 \quad < \quad \theta < \frac{\pi}{2} \quad \text{with} \quad r \sin\alpha \cos\theta = w > 0, \tag{18}$$

$$\frac{\pi}{2} \quad < \quad \theta < \pi \quad \text{with} \quad r \sin\alpha \cos\theta = w < 0 . \tag{19}$$

It corresponds to a positive (or negative) value of w, respectively. We remind the reader that $\sin(\alpha) > 0$ as $\alpha \in]0, \pi[$. In the interval $\theta \in]0, \pi/2[$ (resp. $\theta \in]\pi/2, \pi[$) the centre of mass of particles $1, 2$ and 3 is situated at the right (resp. the left) of the fourth particle. The presence of the centrifugal term in $\beta / \cos(\theta)^2$ forbids the c.m coordinate of particles $(1, 2, 3)$ to coincide with x_4.

The solutions of Eq.(15) in $]0, \pi/2[$ with Dirichlet conditions are well known [14]. The regular solutions are

$$\Theta_{m,n}^{(+)}(\theta) \quad = \quad (\sin\theta)^{b_n + \frac{1}{2}} (\cos\theta)^{c + \frac{1}{2}} P_m^{(b_n, c)}(\cos 2\theta), \tag{20}$$

$$0 \quad \leq \quad \theta \leq \frac{\pi}{2}, \quad m = 0, 1, 2, ..., \quad c = \frac{1}{2}\sqrt{1 + 4\beta} = \frac{1}{2}\sqrt{1 + \frac{g}{3}} . \tag{21}$$

Here, $P_m^{(b_n, c)}$ are the Jacobi polynomials [21]. The index $+$ means that the w-coordinate is positive. The solutions in the interval $\frac{\pi}{2} < \theta < \pi$ are

$$\Theta_{m,n}^{(-)}(\theta) \quad = \quad (\sin\theta)^{b_n + \frac{1}{2}} (-\cos\theta)^{c + \frac{1}{2}} P_m^{(b_n, c)}(\cos 2\theta), \tag{22}$$

$$\frac{\pi}{2} \quad < \quad \theta < \pi, \quad m = 0, 1, 2, ...$$

The index $-$ means the w-coordinate to be negative. Note that in the interval $]\pi/2, \pi[$ we have $-\cos\theta > 0$ and $\sin\theta > 0$ so that the real power of these positive values are defined.

In both cases, the eigenvalues $C_{m,n}$ of Eq.(15) are given by

$$C_{m,n} = (2m + b_n + c + 1)^2, \quad m = 0, 1, 2, ... \tag{23}$$

The third angular equation (10) has been treated in [14]. The regular eigensolutions and corresponding eigenvalues in the interval $]0, \pi[$ read, respectively,

$$G_{\ell,m,n}(\alpha) \quad = \quad (\sin\alpha)^{c_{m,n} + \frac{1}{2}} C_\ell^{(c_{m,n} + \frac{1}{2})}(\cos\alpha), \qquad \ell = 0, 1, 2, ..., \tag{24}$$

$$D_{\ell,m,n} \quad = \quad \left(\ell + c_{m,n} + \frac{1}{2} \right)^2, \qquad\qquad \ell = 0, 1, 2, ... \tag{25}$$

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Here, $c_{m,n} = \sqrt{C_{m,n}}$ (see Eq.(23)). Note that the choice $c_{m,n} > 0$ implies for every value of $\{m,n\}$ the function $G_{\ell,m,n}(\alpha)$ to vanish at the boundaries of the interval $]0,\pi[$.

Finally, the reduced radial equation reads

$$\left(-\frac{d^2}{dr^2} + \omega^2 r^2 + \frac{\mu + D_{\ell,m,n} - \frac{1}{4}}{r^2} - E_{k,\ell,m,n} \right) F_{k,\ell,m,n}(r) = 0. \tag{26}$$

This is nothing but the usual 3-dimensional harmonic oscillator equation, $(\mu + D_{\ell,m,n} - 1/4)/r^2$ replacing the centrifugal barrier. The square integrable solutions are well known, putting a limit on the coefficient of the $1/r^2$ term. Note that taking $\mu + D_{\ell,m,n} = 0$ leads to several self-adjoint extensions differing by a phase. This fact has been discussed in [14]. More details can be found in [23, 24]. It has to be noted that for attractive centrifugal barriers, $\mu + D_{\ell,m,n} < 0$, the problem of collapse appears, unless regularization procedures are taken into account [25, 26, 27, 28]. Using the definition of $D_{\ell,m,n}$, Eq.(25), we have

$$\mu + D_{\ell,m,n} = \mu + (\ell + 2m + 3n + c + 3a + 3)^2 > 0, \quad \forall n \geq 0, \quad \forall m \geq 0, \quad \forall \ell \geq 0. \tag{27}$$

The quantity $\mu + D_{\ell,m,n}$ is minimal for $n = 0, m = 0, \ell = 0$ and $a = c = 0$ (we recall that $a \geq 0$, see (14) and that $c \geq 0$, see (21)). It put constraint on μ, which has to satisfy $\mu > -9$. We introduce the auxiliary parameter $\kappa_{\ell,m,n}$ defined by

$$\kappa_{\ell,m,n}^2 = \mu + D_{\ell,m,n}, \qquad \kappa_{\ell,m,n} = \sqrt{\mu + D_{\ell,m,n}} . \tag{28}$$

The solution of the radial equation (26) vanishing at $r = 0$ and for $r \to \infty$ is

$$F_{k,\ell,m,n}(r) = r^{\kappa_{\ell,m,n} + \frac{1}{2}} \exp\left(-\frac{\omega r^2}{2} \right) L_k^{(\kappa_{\ell,m,n})}(\omega r^2), \qquad k = 0, 1, 2... . \tag{29}$$

The $L_k^{(q)}$ are the generalized Laguerre polynomials [21]. The associated eigenenergies are given by

$$E_{k,\ell,m,n} = 2\omega(2k + \kappa_{\ell,m,n} + 1), \qquad k = 0, 1, 2.... \tag{30}$$

Collecting all pieces, we conclude the physically acceptable solutions of the Schrödinger equation (5) to be given by

$$\begin{aligned}
\Psi_{k,\ell,m,n}(r,\alpha,\theta,\varphi) &= r^{\sqrt{\mu+(\ell+2m+3n+c+3a+3)^2}-1} e^{-\frac{\omega r^2}{2}} L_k^{\left(\sqrt{\mu+(\ell+2m+3n+c+3a+2)^2}\right)}(\omega r^2) \\
&\times (\sin\alpha)^{2m+3n+c+3a+2} C_\ell^{(2m+3n+c+3a+3)}(\cos\alpha) \\
&\times (\sin\theta)^{3n+3a+\frac{3}{2}} (\epsilon_1 \cos\theta)^{c+\frac{1}{2}} P_m^{\left(3n+3a+\frac{3}{2},c\right)}(\cos 2\theta) \\
&\times (\sin 3\varphi)^{a+\frac{1}{2}} C_n^{\left(a+\frac{1}{2}\right)}(\cos 3\varphi), \\
k &= 0,1,2,..., \quad \ell = 0,1,2,..., \quad m = 0,1,2,....,, \quad n = 0,1,2,..., \\
0 \leq \varphi \leq \frac{\pi}{3} , & \quad \frac{1-\epsilon_1}{2}\frac{\pi}{2} \leq \theta \leq \frac{3-\epsilon_1}{2}\frac{\pi}{2}, \epsilon_1 = \pm 1, \quad a = \frac{1}{2}\sqrt{1+2\lambda} , \quad c = \frac{1}{2}\sqrt{1+\frac{g}{3}}
\end{aligned} \tag{31}$$

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Using the prescription of [8] and the parity properties of the Gegenbauer polynomials [21], we can write the solution as a compact form, valid in the whole interval $[0, 2\pi]$ for φ

$$
\begin{aligned}
\Psi_{k,\ell,m,n}(r,\alpha,\theta,\varphi) &= r^{\sqrt{\mu+(\ell+2m+3n+c+3a+3)^2}\,-1} e^{-\frac{\omega r^2}{2}} L_k^{\left(\sqrt{\mu+(\ell+2m+3n+c+3a+2)^2}\right)}(\omega r^2) \\
&\times (\sin\alpha)^{2m+3n+c+3a+2} C_\ell^{(2m+3n+c+3a+3)}(\cos\alpha) \\
&\times (\sin\theta)^{3n+3a+\frac{3}{2}} (\epsilon_1 \cos\theta)^{c+\frac{1}{2}} P_m^{\left(3n+3a+\frac{3}{2},c\right)}(\cos 2\theta) \\
&\times sgn(\sin 3\varphi)^{[(1-\epsilon_2)/2]} |\sin 3\varphi|^{a+\frac{1}{2}} C_n^{\left(a+\frac{1}{2}\right)}(\cos 3\varphi) \ .
\end{aligned}
\tag{32}
$$

Here, $\epsilon_2 = 1$ for bosons and -1 for fermions in equation Eq.(32). We recall that $sgn(x) = x/|x|$ denotes the sign of $x \neq 0$. For the Bose statistics, the extension (32) is possible provided that no δ distribution occurs when the second derivative of the wave function with respect to φ is applied at the boundaries between two adjacent sectors. For example, for $\varphi = \pi/3$ and $n = 0$, a δ distribution occurs for $a = 1/2$ implying $\lambda = 0$. It is due to the presence of $|\sin 3\varphi|$ in (32). As a consequence, the symmetrical solutions for the pure harmonic oscillator ($\lambda = \mu = g = 0$) are not recovered. Also a δ distribution occurs for $c = 1/2$ i.e. $g = 0$ and we do not recover the pure solutions $g = 0$.

The normalization constants $N_{k,\ell,m,n}$ can be calculated from

$$
\int_0^{+\infty} r^3 dr \int_0^\pi \sin^2\alpha \, d\alpha \int_{(1-\epsilon_1)\pi/4}^{(3-\epsilon_1)\pi/4} \sin\theta \, d\theta \int_0^{\frac{\pi}{3}} d\varphi \, \Psi_{k,\ell,m,n}(r,\alpha,\theta,\varphi)\Psi_{k',\ell',m',n'}(r,\alpha,\theta,\varphi)
$$
$$
= \delta_{k,k'}\delta_{\ell,\ell'}\delta_{m,m'}\delta_{n,n'} N_{k,\ell,m,n} \ .
\tag{33}
$$

Use is made, here, of the orthogonality properties of Gegenbauer, Jacobi and Laguerre polynomials [29]. The full expression of the eigenenergies is expressed by

$$
\begin{aligned}
E_{k,\ell,m,n} &\equiv E_{k,\ell+2m+3n} = 2\omega\left(2k+1+\sqrt{\mu+(\ell+2m+3n+c+3a+3)^2}\right), \tag{34} \\
k &= 0,1,2,..., \qquad \ell = 0,1,2,... \qquad m = 0,1,2,... \qquad n = 0,1,2,..., \ .
\end{aligned}
$$

Let us now consider the irregular solutions corresponding to $1/2 - a$. We have to replace a by $-a$ in all equations, from Eq.(12) to Eq.(34). We recall that for $-1/2 < \lambda < 3/2$, these irregular solutions are square integrable as mentioned before. For Fermi statistics, a δ pathology occurs in Eq.(32) for $a = 1/2$ ($\lambda = 0$). Moreover, the requirement of self-adjointness of the Sturm-Liouville operator Eq.(15) imposes us to ensure $b_n \neq 0$ (i.e. to discard the case $\lambda = 0$) and $\beta > -1/4$ ($g > -3$). Consequently, we consider values of λ in $]-1/2, 0[\cup]0, 3/2[$.

Concerning the change $a \mapsto -a$ in the function $\Theta_{m,n}(\theta)$, Eqs.(20,22), the self-adjointness of the Sturm-Liouville operator imposes $B_n \neq 0, \forall n$. Taking into account Eq.(13), we have clearly to look for $a < 1/2$, which is equivalent to $\lambda < 0$. Then we search for which value of a, square integrable solutions $\Theta_{m,n}(\theta)$ are obtained for every values of n. Since

$$
\frac{1}{2} + b_n = 3n + 2 - 3a \quad (a = \frac{1}{2}\sqrt{1+2\lambda}) \ ,
\tag{35}
$$

we have

$$(\forall n \geq 0) \qquad \frac{1}{2} + b_n \geq 2 - 3a \ . \tag{36}$$

The function $\Theta_{m,n}(\theta)$, Eqs.(20,22), leads to square integrable solutions for every n if $a < 5/6$. This last inequality happens for $\lambda < 8/9$. As far as the term $\beta/\cos^2\theta$ is concerned, irregular solutions are found when c is changed to $-c$ in Eqs.(20,21,22). These solutions are square integrable for $c < 1$ $i.e.$ for $\beta < 3/4$ or $g < 9$. Consider the last angular equation concerning the variable α. The differential operator is self-adjoint provided that

$$(\forall m \geq 0) \ (\forall n \geq 0) \qquad c_{m,n} = 2m + 3n + \frac{5}{2} \pm c - 3a \geq \frac{5}{2} - 3a \pm c \ > 0 \ . \tag{37}$$

So that the self-adjointness is ensured when

$$5 > 3\sqrt{1 + 2\lambda} \mp \sqrt{1 + \frac{g}{3}} \ . \tag{38}$$

The square integrability condition for the function $G_{\ell,m,n}$, Eq.(24), requires $c_{m,n} + 1 > 0 \ \forall m \ \forall n$, which yields

$$\pm c - 3a + \frac{7}{2} > 0 \ . \tag{39}$$

This defines an acceptable domain of solutions in λ, g.

As far as the radial equation is concerned, the constraint on $\mu + D_{\ell,m,n} > 0$ is satisfied for $\mu > 0$ otherwise it reads

$$|3 \pm c - 3a| > \sqrt{-\mu} \ . \tag{40}$$

This condition defines a domain of acceptable values of μ depending on the values of λ, g, within the interval $\lambda \in \ ] -1/2, 0[\cup]0, 3/2[$. Under such conditions, the radial solutions, Eq.(29), are square integrable because $\kappa_{\ell,m,n} > 0$.

Note that the spectrum for irregular solutions has eigenvalues lower than the ones corresponding to the regular solutions. They are given by

$$E_{k,\ell,m,n}^{(<)} \equiv E_{k,\ell+2m+3n}^{(<)} = 2\omega \left(2k + 1 + \sqrt{\mu + (\ell + 2m + 3n \pm c - 3a + 3)^2} \right), \tag{41}$$

$$k = 0, 1, 2, ..., \qquad \ell = 0, 1, 2, ... \qquad m = 0, 1, 2, ... \qquad n = 0, 1, 2, ..., \ .$$

2.2 The case of Coulomb-type interaction

In their work on solvable three-body problems in $D = 1$, Khare and Badhuri [11] have considered an attractive interaction of the form $-const./\sqrt{\sum_{i<j}(x_i - x_j)^2}$. In spherical coordinates, such a term leads to an attractive $1/r$ potential. It can be used to replace the harmonic field in Hamiltonian like (1) and produces Coulomb-like radial wave functions. Following this idea we consider the Hamiltonian

$$H = -\sum_{i=1}^{4} \frac{\partial^2}{\partial x_i^2} - \frac{\eta}{\sqrt{\sum_{i=1}^{4} x_i^2}} + \lambda \sum_{i<j}^{3} \frac{1}{(x_i - x_j)^2} + \frac{g}{(x_1 + x_2 + x_3 - 3x_4)^2} + \frac{\mu}{\sum_{i=1}^{4} x_i^2}, \quad \eta > 0 \tag{42}$$

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The same coordinates transformations as Eq.(2) and Eq.(4) are applied to equation Eq.(42), which becomes

$$
\left\{ -\frac{\partial^2}{\partial r^2} - \frac{3}{r}\frac{\partial}{\partial r} - \frac{\eta}{r} + \frac{\mu}{r^2} + \frac{1}{r^2}\left[-\frac{\partial^2}{\partial \alpha^2} - 2\cot\alpha\frac{\partial}{\partial \alpha} + \frac{1}{\sin^2\alpha}\left(-\frac{\partial^2}{\partial \theta^2} - \cot\theta\frac{\partial}{\partial \theta} \right.\right.\right.
$$
$$
\left.\left.\left. + \frac{g}{12\cos^2\theta} + \frac{1}{\sin^2\theta}\left(-\frac{\partial^2}{\partial \varphi^2} + \frac{9\lambda}{2\sin^2(3\varphi)} \right) \right) \right] \right\} \Psi(r,\alpha,\theta,\varphi) = E\Psi(r,\alpha,\theta,\varphi) . \tag{43}
$$

Here, $\Psi(r,\alpha,\theta,\varphi)$ are the eigensolutions associated to eigenenergy E. Compared to Eq.(5), the harmonic confinement has been replaced by a Coulomb-like field.

The resolution of Eq.(43) is achieved in a similar manner as in the preceding section. The solutions of the angular parts are identical. Only the radial part is different. Using the same factorization of the wave function as the one of Eq.(7), we obtain four decoupled differential equations. The eigensolutions and eigenvalues of the angular differential equations are given respectively by Eqs.(12,13) for the variable φ, by Eqs.(20,22) for the variable θ and by Eqs.(24,25) for the variable α.

The reduced radial equation reads

$$
\left(-\frac{d^2}{dr^2} - \frac{\eta}{r} + \frac{\mu + D_{\ell,m,n} - \frac{1}{4}}{r^2} - E_{k,\ell,m,n} \right) F_{k,\ell,m,n}(r) = 0. \tag{44}
$$

As in the case of the harmonic oscillator, the solutions for bound states (negative energies) are well known. The eigenfunctions are given by [30, 31]

$$
\begin{aligned}
F_{k,\ell,m,n}(r) &= r^{\kappa_{\ell,m,n}+\frac{1}{2}}\exp\left(-\tilde{\eta}\,r\right)L_k^{(2\kappa_{\ell,m,n})}(2\tilde{\eta}r) \\
&\quad k = 0,1,2..., \quad \tilde{\eta} = \frac{\eta}{1+2k+2\kappa_{\ell,m,n}} \; .
\end{aligned} \tag{45}
$$

The associated eigenenergies take the form

$$
E_{k,\ell,m,n} = -\frac{4\eta^2}{(2k+\kappa_{\ell,m,n}+1)^2}, \qquad k = 0,1,2.... \tag{46}
$$

with $\kappa_{\ell,m,n}$ given by Eq.(28).

In addition to the bound states, the equation Eq.(44) admits scattering states corresponding to positive energies. They are not studied in this paper.

3   A five-body problem with harmonic confinement.

We consider the Hamiltonian

$$
H = \sum_{i=1}^{5}\left(-\frac{\partial^2}{\partial x_i^2} + \omega^2 x_i^2 \right) + \frac{\mu}{\sum_{i=1}^{5} x_i^2}
$$

$$+ \quad \lambda \sum_{i<j}^{3} \frac{1}{(x_i - x_j)^2} + \frac{\kappa}{(x_4 - x_5)^2} + \frac{g}{[2(x_1 + x_2 + x_3) - 3(x_4 + x_5)]^2} \tag{47}$$

The first contribution gives the energy of the five independent particles in an harmonic field. Residual interactions are added in a following way. The first three particles, with coordinates x_1, x_2, x_3, constituting the first cluster, interact pairwise by a two-body inverse square potentials, of Calogero type [8]. The remaining two-particles, whose coordinates are x_4, x_5, constituting the second cluster, interact also via a two-body inverse square Calogero potential. The centre of mass of both clusters interact via the translationally invariant term $g/(2(x_1 + x_2 + x_3) - 3(x_4 + x_5))^2$. An additional five-body potential, non translationally invariant, represented by the term $\mu/(x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2)$, is added to the whole Hamiltonian.

In order to solve this five-body problem, let us introduce the following coordinates transformation

$$t \;=\; \frac{1}{\sqrt{5}}(x_1 + x_2 + x_3 + x_4 + x_5), \quad u = \frac{1}{\sqrt{2}}(x_1 - x_2), \quad v = \frac{1}{\sqrt{6}}(x_1 + x_2 - 2x_3)$$

$$z \;=\; \frac{1}{\sqrt{2}}(x_4 - x_5), \quad w = \frac{1}{\sqrt{30}}(2(x_1 + x_2 + x_3) - 3(x_4 + x_5)). \tag{48}$$

The transformed Hamiltonian reads:

$$
\begin{aligned}
H \;=\; & -\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial w^2} - \frac{\partial^2}{\partial z^2} - \frac{\partial^2}{\partial u^2} - \frac{\partial^2}{\partial v^2} + \omega^2(t^2 + u^2 + v^2 + z^2 + w^2) \\
& + \frac{9\lambda(u^2 + v^2)^2}{2\,(u^3 - 3uv^2)^2} + \frac{\kappa}{2z^2} + \frac{g}{30\,w^2}.
\end{aligned} \tag{49}
$$

This Hamiltonian is not separable in $\{t, u, v, w, z\}$ variables. As before, we introduce hyperspherical coordinates :

$$
\begin{aligned}
t \;&=\; r\cos\alpha, && w = r\sin\alpha\cos\theta \\
0 \;&\leq\; r < \infty && 0 \leq \alpha \leq \pi \\
z \;&=\; r\sin\alpha\sin\theta\cos\beta, && u = r\sin\alpha\sin\theta\sin\beta\sin\varphi, && v = r\sin\alpha\sin\theta\sin\beta\cos\varphi, \\
0 \;&\leq\; \theta \leq \pi, && 0 \leq \beta \leq \pi, && 0 \leq \varphi \leq 2\pi.
\end{aligned} \tag{50}
$$

The Schrödinger equation is then written:

$$
\begin{aligned}
\Bigg\{ & -\frac{\partial^2}{\partial r^2} - \frac{4}{r}\frac{\partial}{\partial r} + \omega^2 r^2 + \frac{\mu}{r^2} + \frac{1}{r^2}\Bigg[-\frac{\partial^2}{\partial\alpha^2} - 3\cot\alpha\frac{\partial}{\partial\alpha} + \frac{1}{\sin^2\alpha}\bigg(-\frac{\partial^2}{\partial\theta^2} - 2\cot\theta\frac{\partial}{\partial\theta} \\
& + \frac{g}{30\cos^2\theta} + \frac{1}{\sin^2\theta}\bigg(-\frac{\partial^2}{\partial\beta^2} - \cot\beta\frac{\partial}{\partial\beta} + \frac{\kappa}{2\cos^2\beta} + \\
& \frac{1}{\sin^2\beta}\bigg(-\frac{\partial^2}{\partial\varphi^2} + \frac{9\lambda}{2\sin^2(3\varphi)} \bigg)\bigg)\bigg) \Bigg]\Bigg\} \Psi(r,\alpha,\theta,\beta,\varphi) = E\Psi(r,\alpha,\theta,\beta,\varphi) \ .
\end{aligned} \tag{51}
$$

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This Hamiltonian may be mapped to the problem of one particle in the five dimensional space with a non central potential of the form

$$V(r, \alpha, \theta, \varphi) = f_1(r) + \frac{1}{r^2 \sin^2 \alpha} \left[f_2(\theta) + \frac{1}{\sin^2 \theta} \left(f_3(\beta) + \frac{f_4(\varphi)}{\sin^2 \beta} \right) \right]. \tag{52}$$

As in the previous section, the problem becomes separable but in the five variables $\{r, \alpha, \theta, \beta, \varphi\}$. To find the solution we factorize the wave function as follows :

$$\Psi_{k,\ell,j,m,n}(r, \alpha, \theta, \beta, \varphi) = \frac{F_{k,\ell,j,m,n}(r)}{r^2} \frac{G_{\ell,j,m,n}(\alpha)}{\sin^{3/2} \alpha} \frac{\Theta_{j,m,n}(\theta)}{\sin \theta} \frac{H_{m,n}(\beta)}{\sqrt{\sin \beta}} \Phi_n(\varphi). \tag{53}$$

Accordingly, equation (51) separates in five decoupled differential equations:

$$\left(-\frac{d^2}{d\varphi^2} + \frac{9\lambda}{2 \sin^2(3\varphi)} \right) \Phi_n(\varphi) = B_n \Phi_n(\varphi), \tag{54}$$

$$\left(-\frac{d^2}{d\beta^2} + \frac{(B_n - \frac{1}{4})}{\sin^2 \beta} + \frac{\kappa}{2 \cos^2 \beta} \right) H_{m,n}(\beta) = C_{m,n} H_{m,n}(\beta), \tag{55}$$

$$\left(-\frac{d^2}{d\theta^2} + \frac{(C_{m,n} - \frac{1}{4})}{\sin^2 \theta} + \frac{g}{30 \cos^2 \theta} \right) \Theta_{j,m,n}(\theta) = D_{j,m,n} \Theta_{j,m,n}(\theta), \tag{56}$$

$$\left(-\frac{d^2}{d\alpha^2} + \frac{D_{j,m,n} - \frac{1}{4}}{\sin^2 \alpha} \right) G_{\ell,j,m,n}(\alpha) = A_{\ell,j,m,n} G_{\ell,j,m,n}(\alpha), \tag{57}$$

and

$$\left(-\frac{d^2}{dr^2} + \omega^2 r^2 + \frac{\mu + A_{\ell,j,m,n} - \frac{1}{4}}{r^2} \right) F_{k,\ell,j,m,n}(r) = E_{k,\ell,j,m,n} \, F_{k,\ell,j,m,n}(r) \, . \tag{58}$$

The regular solutions of equation (54) on the interval $[0, \pi/3]$, with Dirichlet conditions at the boundaries, are given by the expressions (see above section)

$$\Phi_n(\varphi) = (\sin 3\varphi)^{\frac{1}{2}+a} C_n^{(\frac{1}{2}+a)}(\cos 3\varphi), \quad 0 \leq \varphi \leq \frac{\pi}{3}, \qquad n = 0, 1, 2, ... \tag{59}$$

and associated to the eigenvalues

$$b_n^2 = B_n, \quad b_n = \sqrt{B_n} = \left(3n + 3a + \frac{3}{2} \right), n = 0, 1, 2, ... \quad a = \frac{1}{2}\sqrt{1 + 2\lambda} \tag{60}$$

The second angular equation for the polar angle β reads :

$$\left(-\frac{d^2}{d\beta^2} + \frac{b_n^2 - \frac{1}{4}}{\sin^2 \beta} + \frac{\kappa}{2 \cos^2 \beta} - C_{m,n} \right) H_{m,n}(\beta) = 0 \, . \tag{61}$$

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We have $b_n \neq 0$ so that the Hamiltonian of Eq.(61) is a self-adjoint operator for $\kappa > -1/2$.

The regular solutions of Eq.(61) in $]0, \pi/2[$ with Dirichlet conditions are

$$H_{m,n}^{(+)}(\beta) = (\sin \beta)^{b_n + \frac{1}{2}} (\cos \beta)^{c + \frac{1}{2}} P_m^{(b_n, c)}(\cos 2\beta), \tag{62}$$

$$0 \leq \beta \leq \frac{\pi}{2}, \quad m = 0, 1, 2, ..., \quad c = \frac{1}{2}\sqrt{1 + 2\kappa}, \tag{63}$$

in terms of the Jacobi polynomials. The solutions in the interval $\frac{\pi}{2} < \beta < \pi$ are

$$H_{m,n}^{(-)}(\beta = (\sin \beta)^{b_n + \frac{1}{2}} (-\cos \beta)^{c + \frac{1}{2}} P_m^{(b_n, c)}(\cos 2\beta), \tag{64}$$

$$\frac{\pi}{2} < \beta < \pi, \quad m = 0, 1, 2, ...$$

The index $+$ (resp.$-$) means that the z-coordinate is positive. (resp. negative). Note that in the interval $]\pi/2, \pi[$, we have $-\cos \beta > 0$ and $\sin \beta > 0$, so that the real power of these positive values are defined.

In both cases, the eigenvalues $C_{m,n}$ of Eq.(61) are given by

$$C_{m,n} = c_{m,n}^2, \qquad c_{m,n} = (2m + b_n + c + 1), \quad m = 0, 1, 2, ... \tag{65}$$

The third angular equation for the polar angle θ can be written as

$$\left(-\frac{d^2}{d\theta^2} + \frac{c_{m,n}^2 - \frac{1}{4}}{\sin^2 \theta} + \frac{g}{30 \cos^2 \theta} - D_{j,m,n} \right) \Theta_{j,m,n}(\theta) = 0, \tag{66}$$

Since $c_{m,n} \neq 0$, (see Eqs.(60,65)) the Hamiltonian of Eq.(66) is a self-adjoint operator provided that $g > -15/2$.

The regular solutions Eq.(66) in $]0, \pi/2[$ with Dirichlet conditions read

$$\Theta_{j,m,n}^{(+)}(\theta) = (\sin \theta)^{c_{m,n} + \frac{1}{2}} (\cos \theta)^{d + \frac{1}{2}} P_m^{(c_{m,n}, d)}(\cos 2\theta), \tag{67}$$

$$0 \leq \theta \leq \frac{\pi}{2}, \quad m = 0, 1, 2, ..., \quad d = \frac{1}{2}\sqrt{1 + \frac{2g}{15}}. \tag{68}$$

The solutions in the interval $\frac{\pi}{2} < \theta < \pi$ are

$$\Theta_{j,m,n}^{(-)}(\theta) = (\sin \theta)^{c_{m,n} + \frac{1}{2}} (-\cos \theta)^{d + \frac{1}{2}} P_j^{(c_{m,n}, d)}(\cos 2\theta), \tag{69}$$

$$\frac{\pi}{2} < \theta < \pi, \quad m = 0, 1, 2, ...$$

The index $+$ (resp.$-$) means the w-coordinate to be positive. (resp. negative). Note that in the interval $]\pi/2, \pi[$, we have $-\cos \theta > 0$ and $\sin \theta > 0$, so that the real power of these positive values are defined.

In both cases, the eigenvalues $D_{j,m,n}$ of Eq.(66) are given by

$$D_{j,m,n} = d_{j,m,n}^2 \quad d_{j,m,n} = (2j + c_{m,n} + d + 1), \quad j = 0, 1, 2, ... \tag{70}$$

The regular eigensolutions and corresponding eigenvalues of equation (57) in the interval $]0, \pi[$ read, respectively,

$$G_{\ell,j,m,n}(\alpha) = (\sin \alpha)^{d_{j,m,n}+\frac{1}{2}} C_\ell^{(d_{j,m,n}+\frac{1}{2})}(\cos \alpha), \qquad \ell = 0, 1, 2, ..., \tag{71}$$

$$A_{\ell,j,m,n} = a_{\ell,j,m,n}^2, \quad a_{\ell,j,m,n} = \left(\ell + d_{j,m,n} + \frac{1}{2}\right) \qquad \ell = 0, 1, 2, ... \tag{72}$$

Note that the choice $d_{j,m,n} > 0$ implies for every value of $\{j, m, n\}$ the function $G_{\ell,j,m,n}(\alpha)$ to vanish at the boundaries of the interval $]0, \pi[$.

Finally, the reduced radial equation reads

$$\left(-\frac{d^2}{dr^2} + \omega^2 r^2 + \frac{\mu + A_{\ell,j,m,n} - \frac{1}{4}}{r^2} - E_{k,\ell,j,m,n}\right) F_{k,\ell,j,m,n}(r) = 0. \tag{73}$$

Taking into account the definition of $A_{\ell,j,m,n}$, Eq.(72), we have

$$\mu + A_{\ell,j,m,n} = \mu + (\ell + 2j + 2m + 3n + c + d + 3a + 4)^2 > 0 \quad \forall n \geq 0, \forall m \geq 0, \forall \ell \geq 0 , \tag{74}$$

for every positive μ. The quantity $\mu + A_{\ell,j,m,n}$ is minimal for $n = 0, j = 0, m = 0, \ell = 0$ and $a = 0$ (we recall that $a \geq 0$, see (60) and that $c \geq 0$, see (63)). It put constraint on negative values of μ, namely $-16 < \mu \leq 0$.

We introduce the auxiliary parameter $\kappa_{\ell,j,m,n}$ defined by

$$\kappa_{\ell,j,m,n}^2 = \mu + A_{\ell,j,m,n} , \qquad \kappa_{\ell,j,m,n} = \sqrt{\mu + A_{\ell,j,m,n}} . \tag{75}$$

The solution of the radial equation (73) is

$$F_{k,\ell,j,m,n}(r) = r^{\kappa_{\ell,jm,n}+\frac{1}{2}} \exp\left(-\frac{\omega r^2}{2}\right) L_k^{(\kappa_{\ell,j,m,n})}(\omega r^2), \qquad k = 0, 1, 2... . \tag{76}$$

in terms of the generalized Laguerre polynomials. The associated eigenenergies are given by

$$E_{k,\ell,j,m,n} = 2\omega(2k + \kappa_{\ell,j,m,n} + 1), \qquad k = 0, 1, 2.... \tag{77}$$

Collecting all pieces, we conclude the physically acceptable regular solutions of the Schrödinger equation (51) to be given in a compact form (as explained in the previous section) by

$$
\begin{aligned}
\Psi_{k,\ell,j,m,n}(r,\alpha,\theta,\beta,\varphi) \;=\; & r^{\sqrt{\mu+(\ell+2j+2m+3n+c+d+3a+4)^2}\,-3/2}e^{-\frac{\omega r^2}{2}}L_k^{\left(\sqrt{\mu+(\ell+2j+2m+3n+c+d+3a+3)^2}\right)}(\omega r^2)\\
& \times(\sin\alpha)^{2j+2m+3n+c+d+3a+\frac{5}{2}}C_\ell^{(2j+2m+3n+c+d+4)}(\cos\alpha)\\
& \times(\sin\theta)^{2m+3n+3a+c+2}(\epsilon_2\cos\theta)^{d+\frac{1}{2}}P_j^{\left(2m+3n+c+3a+\frac{5}{2},d\right)}(\cos2\theta)\\
& \times(\sin\beta)^{3n+3a+\frac{3}{2}}(\epsilon_1\cos\beta)^{c+\frac{1}{2}}P_m^{\left(3n+3a+\frac{3}{2},d\right)}(\cos2\beta)\\
& \times sgn(\sin3\varphi)^{[(1-\epsilon_3)/2]}|\sin3\varphi|^{a+\frac{1}{2}}C_n^{\left(a+\frac{1}{2}\right)}(\cos3\varphi),\qquad(78)\\
k \;=\; & 0,1,2,...,\quad \ell=0,1,2,...,\quad j=0,1,2,..,\quad m=0,1,2,..,\quad n=0,1,2,...,\\
& 0\le\varphi\le\frac{\pi}{3}\;,\quad \frac{1-\epsilon_1}{2}\frac{\pi}{2}\le\beta\le\frac{3-\epsilon_1}{2}\frac{\pi}{2},\epsilon_1=\pm1\\
& \frac{1-\epsilon_2}{2}\frac{\pi}{2}\le\theta\le\frac{3-\epsilon_2}{2}\frac{\pi}{2},\epsilon_2=\pm1,\qquad 0\le\alpha\le\pi\\
& a=\frac{1}{2}\sqrt{1+2\lambda}\;,\quad c=\frac{1}{2}\sqrt{1+2\kappa}\quad d=\frac{1}{2}\sqrt{1+\frac{2g}{15}}\;.
\end{aligned}
$$

Here, $\epsilon_3=1$ for bosons and -1 for fermions in equation Eq.(78). For the Bose statistics, the extension (78) is possible provided that no δ distribution appears when the second derivative of the wave function is applied at the boundaries between two adjacent sectors. For $\varphi=\pi/3$ and $n=0$, a δ distribution occurs for $a=1/2$ implying $\lambda=0$, due to the term $|\sin3\varphi|$ in (78). Also a δ distribution occurs for $d=1/2$ i.e. $g=0$ at $\theta=\pi/2$. Consequently, the symmetrical solutions for the pure harmonic oscillator ($\lambda=\mu=g=0$) are not recovered.

The normalization constants $N_{k,\ell,j,m,n}$ can be calculated from

$$
\int_0^{+\infty}r^4dr\int_0^{\pi}\sin^3\alpha\;d\alpha\int_{(1-\epsilon_2)\pi/4}^{(3-\epsilon_2)\pi/4}\sin^2\theta\;d\theta\int_{(1-\epsilon_1)\pi/4}^{(3-\epsilon_1)\pi/4}\sin\beta\;d\beta
$$
$$
\times\int_0^{\frac{\pi}{3}}d\varphi\,\Psi_{k,\ell,j,m,n}(r,\alpha,\theta,\beta,\varphi)\Psi_{k',\ell',j',m',n'}(r,\alpha,\theta,\beta,\varphi)=\delta_{k,k'}\delta_{\ell,\ell'}\delta_{j,j'}\delta_{m,m'}\delta_{n,n'}N_{k,\ell,j,m,n}\;.(79)
$$

As above, use is made of the orthogonality properties of Gegenbauer, Jacobi and Laguerre polynomials.

The full expression of the eigenenergies is expressed by

$$
\begin{aligned}
E_{k,\ell,j,m,n}\;\equiv\;& E_{k,\ell+2j+2m+3n}=2\omega\left(2k+1+\sqrt{\mu+(\ell+2j+2m+3n+c+d+3a+4)^2}\right)\\
k\;=\;& 0,1,2,...,\quad \ell=0,1,2,...\quad j=0,1,2,\quad m=0,1,2,...\quad n=0,1,2,...\;.\qquad(80)
\end{aligned}
$$

4 A six-body problem with harmonic confinement.

We consider the Hamiltonian

$$
\begin{aligned}
H &= \sum_{i=1}^{6}\left(-\frac{\partial^2}{\partial x_i^2}+\omega^2 x_i^2\right)+\lambda_1\sum_{i<j}^{3}\frac{1}{(x_i-x_j)^2}+\lambda_2\sum_{i<j=4}^{6}\frac{1}{(x_i-x_j)^2}\\
&+\frac{g}{(x_1+x_2+x_3-x_4-x_5-x_6)^2}+\frac{\mu}{\sum_{i=1}^{6}x_i^2}
\end{aligned}
\tag{81}
$$

The first contribution gives the energy of the six independent particles in an harmonic field. The first three particles, whose coordinates are x_1, x_2, x_3, constitute the first cluster and the remaining three particles, whose coordinates are x_4, x_5, x_6, constitute the second cluster. In each cluster, the particles interact pairwise by a two-body inverse square potentials. The center of mass of both clusters interact via the six-body translationally invariant inverse square potential, namely $g/(x_1+x_2+x_3-x_4-x_5-x_6)^2$. A non-translationaly invariant six-body potential with coupling constant μ is added.

In order to solve this six-body problem, let us introduce the following coordinates transformation

$$
t = \frac{1}{\sqrt{6}}(x_1+x_2+x_3+x_4+x_5+x_6), \quad u_1=\frac{1}{\sqrt{2}}(x_1-x_2), \quad v_1=\frac{1}{\sqrt{6}}(x_1+x_2-2x_3)
$$

$$
u_2 = \frac{1}{\sqrt{2}}(x_4-x_5), v_2=\frac{1}{\sqrt{6}}(x_4+x_5-2x_6), \quad w=\frac{1}{\sqrt{6}}(x_1+x_2+x_3-x_4-x_5-x_6)
\tag{82}
$$

The transformed Hamiltonian reads:

$$
\begin{aligned}
H &= -\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial w^2}-\frac{\partial^2}{\partial u_1^2}-\frac{\partial^2}{\partial u_2^2}-\frac{\partial^2}{\partial v_1^2}-\frac{\partial^2}{\partial v_2^2}+\omega^2(t^2+w^2+u_1^2+u_2^2+v_1^2+v_2^2)\\
&+\frac{\mu}{t^2+w^2+u_1^2+u_2^2+v_1^2+v_2^2}+\frac{9\lambda_1(u_1^2+v_1^2)^2}{2\left(u_1^3-3u_1v_1^2\right)^2}+\frac{9\lambda_2(u_2^2+v_2^2)^2}{2\left(u_2^3-3u_2v_2^2\right)^2}+\frac{g}{6\,w^2}.
\end{aligned}
\tag{83}
$$

This Hamiltonian is not separable in $\{t, u_1, v_1, u_2, v_2, w\}$ variables. We first introduce the following coordinates transformation :

$$
\begin{aligned}
u_1 &= r_1\sin\varphi_1, \quad v_1=r_1\cos\varphi_1, \quad 0\le r_1<\infty, \quad 0\le\varphi_1\le 2\pi\\
u_2 &= r_2\sin\varphi_2, \quad v_2=r_2\cos\varphi_2, \quad 0\le r_2<\infty, \quad 0\le\varphi_2\le 2\pi
\end{aligned}
\tag{84}
$$

The Schrödinger equation is then written:

$$
\begin{aligned}
\Bigg\{&-\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial w^2}-\frac{\partial^2}{\partial r_1^2}-\frac{1}{r_1}\frac{\partial}{\partial r_1}-\frac{\partial^2}{\partial r_2^2}-\frac{1}{r_2}\frac{\partial}{\partial r_2}+\frac{g}{6w^2}+\omega^2(r_1^2+r_2^2+t^2+w^2)\\
&+\frac{\mu}{r_1^2+r_2^2+t^2+w^2}+\frac{1}{r_1^2}\left[-\frac{\partial^2}{\partial\varphi_1^2}+\frac{9\lambda_1}{2\sin^2(3\varphi_1)}\right]\\
&+\frac{1}{r_2^2}\left[-\frac{\partial^2}{\partial\varphi_2^2}+\frac{9\lambda_2}{2\sin^2(3\varphi_2)}\right]\Bigg\}\Psi(t,w,r_1,r_2,\varphi_1,\varphi_2)=E\Psi(t,w,r_1,r_2,\varphi_1,\varphi_2)\ .
\end{aligned}
\tag{85}
$$

The potential involved in the equation (85)

$$V(t, w, r_1, r_2, \varphi_1, \varphi_2) \;=\; \omega^2(r_1^2 + r_2^2 + t^2 + w^2) + \frac{\mu}{r_1^2 + r_2^2 + t^2 + w^2} + \frac{g}{6w^2}$$
$$+ \;\; \frac{1}{r_1^2}\,\frac{9\lambda_1}{2\sin^2(3\varphi_1)} + \frac{1}{r_2^2}\,\frac{9\lambda_2}{2\sin^2(3\varphi_2)} \tag{86}$$

has the general form

$$V(t, w, r_1, r_2, \varphi_1, \varphi_2) = f(t, w, r_1, r_2) + \frac{f_1(\varphi_1)}{r_1^2} + \frac{f_2(\varphi_2)}{r_2^2} \; . \tag{87}$$

This suggests the wave function to be factorized as follows

$$\Psi(t, w, r_1, r_2, \varphi_1, \varphi_2) = \tilde{\Psi}(t, w, r_1, r_2) \prod_{j=1}^{2} \Phi_j(\varphi_j) \; . \tag{88}$$

The equation (85) will be solved in two steps. Firstly we solve

$$\left(-\frac{d^2}{d\varphi_1^2} + \frac{9\lambda_1}{2\sin^2(3\varphi_1)} \right) \Phi_{n_1}^{(1)}(\varphi_1) = B_{n_1}^{(1)}\Phi_{n_1}^{(1)}(\varphi_1), \tag{89}$$

and

$$\left(-\frac{d^2}{d\varphi_2^2} + \frac{9\lambda_2}{2\sin^2(3\varphi_2)} \right) \Phi_{n_2}^{(2)}(\varphi_2) = B_{n_2}^{(2)}\Phi_{n_2}^{(2)}(\varphi_2), \tag{90}$$

on the interval $[0, \pi/3]$, with Dirichlet conditions at the boundaries. The $B_{n_j}^{(j)}, j = 1, 2$ are the quantified eigenvalues of the equations (89,90), respectively given by

$$B_{n_j}^{(j)} = (b_{n_j}^{(j)})^2 \qquad b_{n_j}^{(j)} = 3\left(n_j + \frac{1}{2} + a_j \right), \qquad n_j = 0, 1, 2, \dots \quad j = 1, 2 \; . \tag{91}$$

The associated eigensolutions are given in terms of the Gegenbauer polynomials $C_n^{(q)}$

$$\Phi_{n_j}^{(j)}(\varphi_j) = (\sin 3\varphi_j)^{\frac{1}{2}+a_j} C_{n_j}^{(\frac{1}{2}+a_j)}(\cos 3\varphi_j), \quad 0 \le \varphi \le \frac{\pi}{3}, \qquad n_j = 0, 1, 2, .., \quad a_j = \frac{1}{2}\sqrt{1 + 2\lambda_j} \; . \tag{92}$$

The second step consists in the resolution of the following Schrödinger equation

$$\left\{ -\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial w^2} - \frac{\partial^2}{\partial r_1^2} - \frac{1}{r_1}\frac{\partial}{\partial r_1} - \frac{\partial^2}{\partial r_2^2} - \frac{1}{r_2}\frac{\partial}{\partial r_2} + \frac{g}{6w^2} \right.$$
$$+ \omega^2(r_1^2 + r_2^2 + t^2 + w^2) + \frac{\mu}{r_1^2 + r_2^2 + t^2 + w^2}$$
$$\left. + \frac{B_{n_1}^{(1)}}{r_1^2} + \frac{B_{n_2}^{(2)}}{r_2^2} \right\} \tilde{\Psi}_{n_1,n_2}(t, w, r_1, r_2) = E_{n_1,n_2}\tilde{\Psi}_{n_1,n_2}(t, w, r_1, r_2). \tag{93}$$

This Hamiltonian is not separable in $\{t, w, r_1, r_2\}$ variables. We then introduce the following hyperspherical coordinates :

$$
\begin{aligned}
t &= r\cos\alpha, \quad w = r\sin\alpha\cos\theta, \quad r_1 = r\sin\alpha\sin\theta\sin\beta \quad r_2 = r\sin\alpha\sin\theta\cos\beta \\
0 &\leq r < \infty, \quad\; 0 \leq \alpha \leq \pi, \qquad\quad 0 \leq \theta \leq \pi, \qquad\qquad 0 \leq \beta \leq \pi/2.
\end{aligned}
\tag{94}
$$

Since r_1, r_2 are positive we have $\beta \in [0, \pi/2]$.

The Schrödinger equation (93) becomes :

$$
\begin{aligned}
\Bigg\{ &-\frac{\partial^2}{\partial r^2} - \frac{5}{r}\frac{\partial}{\partial r} + \omega^2 r^2 + \frac{\mu}{r^2} + \frac{1}{r^2}\Bigg[-\frac{\partial^2}{\partial\alpha^2} - 4\cot\alpha\frac{\partial}{\partial\alpha} \\
&+ \frac{1}{\sin^2\alpha}\left(-\frac{\partial^2}{\partial\theta^2} - 3\cot\theta\frac{\partial}{\partial\theta} + \frac{g}{6\cos^2\theta} + \frac{1}{\sin^2\theta}\left(-\frac{\partial^2}{\partial\beta^2} - 2\cot 2\beta\frac{\partial}{\partial\beta}\right.\right. \\
&\left.\left.\frac{B_{n_1}^{(1)}}{\sin^2\beta} + \frac{B_{n_2}^{(2)}}{\cos^2\beta}\right)\right)\Bigg]\Bigg\} \tilde{\Psi}_{n_1,n_2}(r,\alpha,\theta,\beta) = E_{n_1,n_2}\tilde{\Psi}_{n_1,n_2}(r,\alpha,\theta,\beta).
\end{aligned}
\tag{95}
$$

This Hamiltonian may be mapped to the problem of one particle in the four dimensional space with a non central potential of the form

$$
V(r,\alpha,\theta,\beta) = f_1(r) + \frac{1}{r^2\sin^2\alpha}\left[f_2(\theta) + \frac{1}{\sin^2\theta}f_3(\beta)\right].
\tag{96}
$$

As in the second section, the problem becomes separable in the four variables $\{r, \alpha, \theta, \beta\}$. To find the solution we factorize the wave function as follows :

$$
\tilde{\Psi}_{n_1,n_2}(r,\alpha,\theta,\beta) = \frac{F_{k,\ell,j,m,n_1,n_2}(r)}{r^{5/2}}\frac{G_{\ell,j,m,n_1,n_2}(\alpha)}{\sin^2\alpha}\frac{\Theta_{j,m,n_1,n_2}(\theta)}{\sin\theta^{3/2}}\frac{H_{m,n_1,n_2}(\beta)}{\sqrt{\sin 2\beta}} .
\tag{97}
$$

Accordingly, equation (95) separates in four decoupled differential equations:

$$
\left(-\frac{d^2}{d\beta^2} + \frac{(b_{n_1}^{(1)})^2 - 1/4}{\sin^2\beta} + \frac{(b_{n_2}^{(2)})^2 - 1/4}{\cos^2\beta}\right) H_{m,n_1,n_2}(\beta) = C_{m,n_1,n_2}H_{m,n_1,n_2}(\beta),
\tag{98}
$$

in terms of the $b_{n_j}^{(j)}$ of Eq.(91).

$$
\left(-\frac{d^2}{d\theta^2} + \frac{(C_{m,n_1,n_2} - \frac{1}{4})}{\sin^2\theta} + \frac{g}{6\cos^2\theta}\right) \Theta_{j,m,n_1,n_2}(\theta) = D_{j,m,n_1,n_2}\Theta_{j,m,n_1,n_2}(\theta),
\tag{99}
$$

$$
\left(-\frac{d^2}{d\alpha^2} + \frac{D_{j,m,n_1,n_2} - \frac{1}{4}}{\sin^2\alpha}\right) G_{\ell,j,m,n_1,n_2}(\alpha) = A_{\ell,j,m,n_1,n_2}G_{\ell,j,m,n_1,n_2}(\alpha),
\tag{100}
$$

and

$$
\left(-\frac{d^2}{dr^2} + \omega^2 r^2 + \frac{\mu + A_{\ell,j,m,n_1,n_2} - \frac{1}{4}}{r^2}\right) F_{k,\ell,j,m,n_1,n_2}(r) = E_{k,\ell,j,m,n_1,n_2} \; F_{k,\ell,j,m,n_1,n_2}(r) .
\tag{101}
$$

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The regular solutions of Eq.(98), for $\beta \in [0, \pi/2]$ (remind that $r_1, r_2 \geq 0$), with Dirichlet conditions at the boundaries read

$$H_{m,n_1,n_2}(\beta) = (\sin\beta)^{b_{n_1}^{(1)}+\frac{1}{2}} (\cos\beta)^{b_{n_2}^{(2)}+\frac{1}{2}} P_m^{(b_{n_1}^{(1)},b_{n_2}^{(2)})}(\cos 2\beta), \tag{102}$$

$$0 \leq \beta \leq \frac{\pi}{2}, \quad m = 0, 1, 2, ..., \tag{103}$$

in terms of the Jacobi polynomials $P_m^{(b_{n_1}^{(1)},b_{n_2}^{(2)})}$. The associated eigenvalues C_{m,n_1,n_2} are given by

$$C_{m,n_1,n_2} = c_{m,n_1,n_2}^2, \qquad c_{m,n_1,n_2} = (2m + b_{n_1}^{(1)} + b_{n_2}^{(2)} + 1), \quad m = 0, 1, 2, ... \tag{104}$$

Since $b_{n_j}^{(j)} \neq 0, j = 1, 2$ the Hamiltonian of Eq.(98) is a self-adjoint operator.

The second angular equation for the polar angle θ reads :

$$\left(-\frac{d^2}{d\theta^2} + \frac{c_{m,n_1,n_2}^2 - 1/4}{\sin^2\theta} + \frac{d^2 - 1/4}{\cos^2\theta} - D_{j,m,n_1,n_2} \right) \Theta_{j,m,n_1,n_2}(\theta) = 0, \tag{105}$$

with c_{m,n_1,n_2} given by Eq.(104) and $d = \sqrt{g/6 + 1/4}$ with the condition that $g \geq -3/2$. Since $c_{m,n_1,n_2} \neq 0$, the Hamiltonian of Eq.(105) is a self-adjoint operator if $d \neq 0$ i.e. $g \neq -3/2$.

The regular solutions of Eq.(105) in $]0, \pi/2[$ with Dirichlet conditions read, in terms of the Jacobi polynomials,

$$\Theta_{j,m,n_1,n_2}^{(+)}(\theta) = (\sin\theta)^{c_{m,n_1,n_2}+\frac{1}{2}}(\cos\theta)^{d+\frac{1}{2}} P_j^{(c_{m,n_1,n_2},d)}(\cos 2\theta), \tag{106}$$

$$0 \leq \theta \leq \frac{\pi}{2}, \quad j = 0, 1, 2, ..., \quad d = \frac{1}{2}\sqrt{1 + \frac{2g}{3}}. \tag{107}$$

The index $+$ means that the w-coordinate is positive. The solutions in the interval $\frac{\pi}{2} < \theta < \pi$ are

$$\Theta_{j,m,n_1,n_2}^{(-)}(\theta) = (\sin\theta)^{c_{m,n_1,n_2}+\frac{1}{2}}(-\cos\theta)^{d+\frac{1}{2}} P_j^{(c_{m,n_1,n_2},d)}(\cos 2\theta), \tag{108}$$

$$\frac{\pi}{2} < \theta < \pi, \quad j = 0, 1, 2, ... \tag{109}$$

The index $-$ means the w-coordinate to be negative. Note that in the interval $]\pi/2, \pi[$ we have $-\cos\theta > 0$ and $\sin\theta > 0$ so that the real power of these positive values are defined.

In both cases, the eigenvalues D_{j,m,n_1,n_2} of Eq.(105) are given by

$$D_{j,m,n_1,n_2} = d_{j,m,n_1,n_2}^2, \quad d_{j,m,n_1,n_2} = (2j + c_{m,n_1,n_2} + d + 1), \quad j = 0, 1, 2, ... \tag{109}$$

The regular eigensolutions and corresponding eigenvalues in the interval $]0, \pi[$ of the third equation (100) read, respectively,

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$$G_{\ell,j,m,n_1,n_2}(\alpha) \;=\; (\sin\alpha)^{d_{j,m,n_1,n_2}+\frac{1}{2}}\, C_\ell^{(d_{j,m,n_1,n_2}+\frac{1}{2})}(\cos\alpha), \qquad \ell = 0,1,2,..., \tag{110}$$

$$A_{\ell,j,m,n_1,n_2} \;=\; a_{\ell,j,m,n_1,n_2}^2, \quad a_{\ell,j,m,n_1,n_2} = \left(\ell + d_{j,m,n_1,n_2} + \frac{1}{2}\right), \qquad \ell = 0,1,2,.... \tag{111}$$

Note that the choice $d_{j,m,n_1,n_2} > 0$ implies for every value of $\{j,m,n_1,n_2\}$ the function $G_{\ell,j,m,n_1,n_2}(\alpha)$ to vanish at the boundaries of the interval $]0,\pi[$.

Finally, the reduced radial equation reads

$$\left(-\frac{d^2}{dr^2} + \omega^2 r^2 + \frac{\mu + A_{\ell,j,m,n_1,n_2} - \frac{1}{4}}{r^2} - E_{k,\ell,j,m,n_1,n_2}\right) F_{k,\ell,j,m,n_1,n_2}(r) = 0. \tag{112}$$

Taking into account the definition of A_{ℓ,j,m,n_1,n_2}, Eq.(111), the constraint

$$\mu + A_{\ell,j,m,n_1,n_2} \;=\; \mu + \left(\ell + 2j + 2m + 3n_1 + 3n_2 + d + 3a_1 + 3a_2 + \frac{11}{2}\right)^2 > 0$$

$$\forall n_1 \geq 0, \quad \forall n_2 \geq 0, \quad \forall m \geq 0, \quad \forall j \geq 0, \quad \forall \ell \geq 0, \tag{113}$$

happens, for every positive value of μ. Since $\mu + A_{\ell,j,m,n}$ is minimal for $n=0, j=0, m=0, \ell=0$ and $a_1 = a_2 = d = 0$, it put constraint on negative values of μ, which has to satisfy

$$-\left(\frac{\sqrt{11}}{2}\right)^2 < \mu \leq 0 \tag{114}$$

We introduce the auxiliary parameter $\kappa_{\ell,j,m,n_1,n_2}$ defined by

$$\kappa_{\ell,j,m,n_1,n_2} = \sqrt{\mu + A_{\ell,j,m,n_1,n_2}} = \sqrt{\mu + \left(\ell + 2j + 2m + 3n_1 + 3n_2 + d + 3a_1 + 3a_2 + \frac{11}{2}\right)^2} \tag{115}$$

The solution of the radial equation (112) vanishing at $r = 0$ and $r \to \infty$ are :

$$F_{k,\ell,j,m,n_1,n_2}(r) = r^{\kappa_{\ell,j,m,n_1,n_2}+\frac{1}{2}} \exp\left(-\frac{\omega r^2}{2}\right) L_k^{(\kappa_{\ell,j,m,n_1,n_2})}(\omega r^2), \qquad k = 0,1,2... \tag{116}$$

in terms of the generalized Laguerre polynomials and the associated eigenenergies are given by

$$E_{k,\ell,j,m,n_1,n_2} = 2\omega(2k + \kappa_{\ell,j,m,n_1,n_2} + 1), \qquad k = 0,1,2.... \tag{117}$$

Collecting all pieces, we conclude the physically acceptable solutions of the Schrödinger equation (85) to be given by

$$\Psi_{k,\ell,j,m,n_1,n_2}(r,\alpha,\theta,\beta,\varphi_1,\,\varphi_2) = \; r^{\kappa_{\ell,j,m,n_1,n_2}-2}\; e^{-\frac{\omega r^2}{2}}\; L_k^{(\kappa_{\ell,j,m,n_1,n_2})}(\omega r^2)$$

$$\times(\sin\alpha)^{2j+2m+3n_1+3n_2+d+3a_1+3a_2+\frac{7}{2}}C_\ell^{(2j+2m+3n_1+3n_2+d+3a_1+3a_2+\frac{11}{2})}(\cos\alpha)$$

$$\times(\sin\theta)^{2m+3n_1+3n_2+3a_1+3a_2+3}(\epsilon_3\cos\theta)^{d+\frac{1}{2}}P_j^{(2m+3n_1+3n_2+3a_1+3a_2+4,d)}(\cos 2\theta)$$

$$\times(\frac{1}{\sqrt{2}})(\sin\beta)^{3n_1+3a_1+\frac{3}{2}}(\cos\beta)^{3n_2+3a_2+\frac{3}{2}}P_m^{(3n_1+3a_1+\frac{3}{2},3n_2+3a_2+\frac{3}{2})}(\cos 2\beta)$$

$$\times sgn(\sin 3\varphi_1)^{[(1-\epsilon)/2]}|\sin 3\varphi_1|^{a_1+\frac{1}{2}}C_{n_1}^{(a_1+\frac{1}{2})}(\cos 3\varphi_1),$$

$$\times sgn(\sin 3\varphi_2)^{[(1-\epsilon)/2]}|\sin 3\varphi_2|^{a_2+\frac{1}{2}}C_{n_2}^{(a_2+\frac{1}{2})}(\cos 3\varphi_2), \tag{118}$$

with $\epsilon = 1$ for bosons and $\epsilon = -1$ for fermions and

$$k \;=\; 0,1,2,...,\ell = 0,1,2,...,j = 0,1,2,..,m = 0,1,2,...,n_1 = 0,1,2,...,n_2 = 0,1,2,...$$

$$0 \;\leq\; \varphi \leq \frac{\pi}{3}\;,0 \leq \beta \leq \frac{\pi}{2}, \frac{1-\epsilon_3}{2}\frac{\pi}{2} \leq \theta \leq \frac{3-\epsilon_3}{2}\frac{\pi}{2},\epsilon_3 = \pm 1, 0 \leq \alpha \leq \pi$$

$$a_1 = \frac{1}{2}\sqrt{1+2\lambda_1}\;,\qquad a_2 = \frac{1}{2}\sqrt{1+2\lambda_2}\;,\quad d = \frac{1}{2}\sqrt{1+\frac{2g}{3}}$$

It has to be noticed that, for Bose statistics, a δ pathology occurs in (118) for $a_j = 1/2$ ($\lambda_j = 0, j = 1, 2$) and $d = 1/2$ ($g = 0$)

The normalization constants N_{k,ℓ,j,m,n_1,n_2} can be calculated from

$$\int_0^{+\infty} r^5 dr \int_0^{\pi}\sin^4\alpha\; d\alpha \int_{(1-\epsilon_3)\pi/4}^{(3-\epsilon_3)\pi/4}\sin^3\theta\;\; d\theta \int_0^{\pi/2}\sin(2\beta)\;d\beta \int_0^{\frac{\pi}{3}}d\varphi_1 \int_0^{\frac{\pi}{3}}d\varphi_2$$

$$\times\Psi_{k,\ell,j,m,n_1 n_2}(r,\alpha,\theta,\beta,\varphi_1,\varphi_2)\Psi_{k',\ell',j',m',n_1',n_2'}(r,\alpha,\theta,\beta,\varphi_1,\varphi_2) = \delta_{k,k'}\delta_{\ell,\ell'}\delta_{j,j'}\delta_{m,m'}\delta_{n_1,n_1'}\delta_{n_2,n_2'}N_{k,\ell,j,m,n_1,n_2}.$$

As above, use is made of the orthogonality properties of Gegenbauer, Jacobi and Laguerre polynomials.

The eigenenergies are

$$E_{k,\ell,j,m,n_1,n_2} = 2\omega\left(1 + 2k + \sqrt{\mu + \left(\ell + 2j + 2m + 3n_1 + 3n_2 + d + 3a_1 + 3a_2 + +\frac{11}{2}\right)^2}\right) \tag{119}$$

Partial degeneracies are observed i.e. all solutions for which $\ell + 2j + 2m + 3n_1 + 3n_2 = N$ is satisfied, N, being fixed.

5   Conclusions

In this paper, we have proposed and exactly solved few-body quantum problems of four, five and six particles in the $D = 1$ dimensional space. The particles are confined in a mean field generated by harmonic or attractive Coulomb-type forces. The interactions between particles are governed by two-body inverse square Calogero potentials with both translationally and non-translationally invariant many-body forces, for the three problems treated. The difficulties encountered concerning the explicit expression of the eigenfunctions, in problems concerned by the pairwise Calogero interaction between each pair of particles, have been overcome by restricting the number of two-body interactions. Our choice singularizes the interaction between one cluster of three particles with one particle for the four-body problem. For the five-body problem , we singularize the interaction between two clusters of two and three particles respectively. Finally, for the six-body problem, we choose the interaction between two clusters each one formed by three particles.

Appropriate coordinates transformations, lead to the solution of the stationary Schödinger equation, by providing the full regular eigensolutions with the eigenenergies spectrum for these few-body bounded systems.

For the four-body problem we pay attention to the irregular solutions. The coupling constants domains for which the irregular solutions, become square integrable are determined. Also for the four-body problem we have replaced the harmonic confinement by an attractive "Coulomb-type" interaction, and the solutions are given for the bound states only. This study can be extended straightforwardly to the five and six-body problems.

The present results suggest to extend the construction of exactly solvable models to larger systems of particles.

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